

What causes bullet's wind drift and how significant is it in pistol shooting?

E. G. Mishchenko

Undergraduate Seminar, Department of Physics and Astronomy, University of Utah

(Dated: adapted from Feb 28, 2013 presentation)

It is well-known that rifle bullets fired over long distances could experience considerable wind drift that an athlete must take into account if she wants to achieve an accurate shot. But how important is the wind drift for a pistol shooter? Free pistol Olympic precision shooting is performed at rather short distances of 50 meters with relatively low bullet velocities, ~ 300 m/s. Interestingly, these two conditions are what makes this problem amenable to a fairly simple “back-of-the-envelope” calculation. But to make a reasonably accurate estimate we need to understand the actual mechanism by which wind makes the bullet to change its path in the first place. Wait, what a silly question! Could the answer be more obvious? The wind surely pushes against the side of the bullet causing its trajectory to deviate from the straight line. Unexpectedly, if this “push” were the only thing to worry about the drift would be fairly negligible. The actual effect is due to a different cause and is not as weak.

We cannot address forces induced by crosswind without first discussing the origins of the aerodynamic drag force. Consider an object, for example a physicist's darling, sphere of diameter d , moving through still air with velocity $\vec{u}_0$. Equivalently, in the reference frame of the object, air flows past it with the velocity $-\vec{u}_0$. At *low velocities* the drag force comes primarily from the air viscosity. Viscosity is intuitively a very simple concept. Right at the surface of the object the air “sticks” to it so that the local velocity is strictly zero, $u = 0$, see Fig. 1. The gradient of the velocity $\partial u / \partial z$ in the direction perpendicular to the surface is finite, however. The viscous force acting along the surface is proportional to that gradient,

$$\frac{F}{A} = \eta \frac{\partial u}{\partial z}, \quad (1)$$

and scales with the area A of the surface; the viscosity coefficient η is a physical characteristics of the substance (gas, fluid). For air $\eta = 1.5 \times 10^{-5} \frac{\text{kg}}{\text{m}\cdot\text{s}}$. The definition (1) is already sufficient to estimate the drag force acting on our sphere. The *only* length scale available is the diameter of the sphere d , so the velocity gradient *has* to be $\partial u / \partial z \sim u_0 / d$, at least by the order of magnitude. The area of the surface of the sphere is $A = \pi d^2$. We can now write for the total drag force

$$F \sim \pi \eta u_0 d. \quad (2)$$

This formula is known as the Stokes law. The missing coefficient can only be determined by the exact solution of the hydrodynamic (Navier-Stokes) equations and is equal to 3. With increasing velocity u the Stokes law fails as viscosity becomes progressively less important compared with the force of the air directly “crashing”

into the object. Here too an order-of-magnitude estimate is simple to perform. Over time Δt the volume $u \Delta t A_{\perp}$ of air will come into contact with the object, where $A_{\perp}$ is its the cross sectional area (for a sphere $A_{\perp} = \pi d^2 / 4$). Correspondingly, multiplying this volume by the air density ρ and velocity u we find the total momentum $\Delta P = \rho A_{\perp} u^2 \Delta t$ of the air that is striking the object. This also gives a rough estimate of the momentum transferred to the object, so that the drag force $F \sim \Delta P / \Delta t \sim \rho A_{\perp} u^2$, again by the order of magnitude. The missing coefficient would account for the fact that not all of the air is completely stopped by the object, so that the actual momentum transferred should be somewhat less than our estimate. This correction is commonly accounted for by introducing a numerical factor

$$F = \frac{C_D}{2} \rho A_{\perp} u^2. \quad (3)$$

in terms of the *drag coefficient* C_D , which of course de-

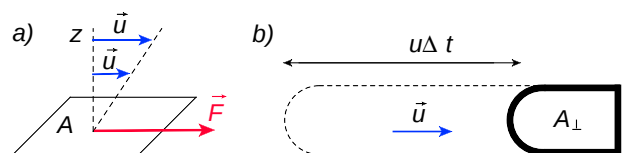


FIG. 1: The mechanisms of drag for different velocities/Reynolds numbers of the flow: a) At small Re viscous force proportional to the velocity gradient near the surface drag dominates; b) at large Re the force is determined (by the order of magnitude) by the amount of momentum carried by the air in a volume $u A_{\perp}$ striking the object per unit of time (in the reference frame of the object).

pends on the shape of the object. For a sphere $C_D = 0.47$ while for a thin disc (moving perpendicular to its surface) C_D raises to 1.1. The formulas (2) and (3) describe the same phenomenon but at different values of the velocity u .

Let us now think at what velocities does the quadratic expression (3) replace the linear Stokes law (2)? It is not difficult to guess that this should happen when both expressions “match”, i.e. become roughly equal to each other. Neglecting numerical factors we see that it occurs when $u \sim \eta / \rho d$. Equivalently, one could construct a dimensionless number, so-called *Reynolds number*,

$$\text{Re} = \frac{\rho d u}{\eta}, \quad (4)$$

and check whether it is small under particular conditions (in which case the Stokes law has to be used) or large (where the other expression is ordered).

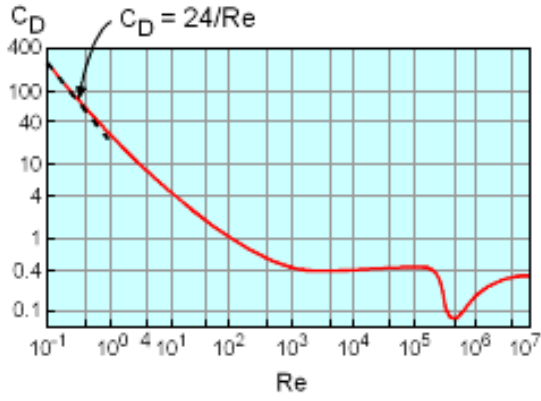


FIG. 2: Drag coefficient $C_D = 2F/\rho A_\perp u^2$ as a function of the Reynolds number $Re = \rho du/\eta$ in as the log-log plot. (This is a classic graph of the fluid mechanics, this particular image taken from: <http://www.ecourses.ou.edu>).

Let me point out that the situations where two limits “cross over” into each other are ubiquitous in physics. In a vast majority of them where two expression should match they do so in a region where their ratio is ~ 1 . Interestingly, the aerodynamic drag is one notable exception, as the “switching” from (2) to (3) occurs at Re of the order of *several hundred*. Fig. 2 illustrates how this crossover happens for the case of a sphere as the Reynolds number is increased (which itself can be viewed as a dimensionless measure of velocity, in units of $\eta/\rho d$). At not very large $Re < 100$ the Stokes law is the right approximation; between $Re \sim 10^2 - 10^3$ it crosses over to the quadratic behavior (3) which remains valid for over two orders of magnitude before a sudden reduction of the drag occurs (the “drag crisis”). The latter is a very interesting phenomenon but well beyond the scope of our discussion.

Let us see where small-bore pistol bullet parameters land us on this graph. Diameter of the bullet $d \sim 0.5$ cm, velocity $v \approx 300$ m/s, and air density $\rho = 1.2$ kg/m³ yield $Re \sim 1.5 \times 10^5$. This number is well within the plateau region of the graph on Fig. 2, where the drag force is described by the formula (3) that from now on is going to be used.

We are now equipped with enough understanding to perform the estimate of the wind drift effect. We assume the most unfavorable situation of a wind blowing exactly 90 degrees to the bullet’s path with the velocity u_{wind} . Repeating the momentum transfer arguments of the preceding paragraphs one would expect that the wind push on the side of the bullet is given by $F_{wind} \sim \rho A u_{wind}^2$, which is *second order* in the wind velocity. However, such naïve reasoning would greatly underestimate the drift effect. An attentive reader can guess that something is amiss after realizing that the amount of air the bullet comes into contact with is in fact determined by the velocity of the bullet u_0 that under any practical conditions is *much greater* than u_{wind} .

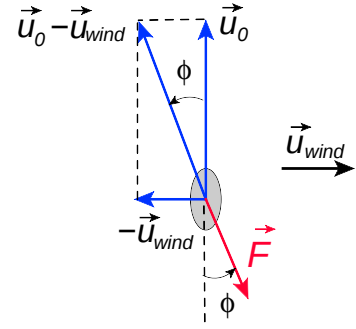


FIG. 3: Finding the drag force acting on the bullet: wind velocity is shown via black arrow. Tilted blue arrow indicates the direction of the bullet’s velocity $\vec{u}_0 - \vec{u}_{wind}$ in the reference frame of the wind, it is given by the vector difference of the bullet’s velocity $\vec{u}_0$ with respect to the ground and the velocity of the wind.

The easiest way to proceed is illustrated with Fig. 3. In the reference frame of the wind the bullet’s velocity is given by the vector $\vec{u}_0 - \vec{u}_{wind}$, which makes a small angle $\phi \approx \tan \phi = u_{wind}/u_0$ with the bullet’s axis. The magnitude of the *total* drag force is changed very insignificantly [1] from its “windless” value $C_D \rho A_\perp u_0^2/2$, but its *direction* is *tilted* away from $\vec{u}_0$ by the angle ϕ . The resulting lateral force is

$$F_x = \frac{C_D}{2} \rho A_\perp u^2 \times \frac{u_{wind}}{u_0} = \frac{C_D}{2} \rho A_\perp u_0 u_{wind}. \quad (5)$$

It is now an exercise in kinematics to find the bullet’s horizontal displacement that occurs over the duration of its flight to the target. The lateral acceleration is $a_x = F_x/m$, which after time t (starting with zero lateral velocity) leads to the horizontal shift $\Delta x = a_x t^2/2$. The time of flight is roughly the ratio of the distance to the target L to the bullet’s muzzle velocity, $t = L/u_0$, if the bullet’s deceleration is disregarded (see below). By putting now everything together we arrive at the estimate,

$$\Delta x = \frac{C}{4} \frac{\rho A_\perp L^2}{m} \frac{u_{wind}}{u_0}. \quad (6)$$

Note, foremost, that the drift effect turned out to be of the *first order* in the wind velocity. Qualitatively our finding should be understood as follows. The lateral wind drift is caused by the *rotation of the total drag force* over the angle proportional to the crosswind speed. This rotation is *kinematic* in origin; in other words it is due to the vector character of velocities that add according to Fig. 3.

We can now apply formula (6) to the Olympic free pistol shooting which is done at distance of $L = 50$ m with the .22lr caliber ammunition (“small-bore”). The most popular bullets have weight of 40 grain or 2.3×10^{-3} kg in the SI system of units. The typical bullet’s muzzle velocity is slightly subsonic, $u_0 \approx 300$ m/s, and the bullet’s diameter (as explicitly indicated in its caliber) is $d = 0.224$

inch = 0.56 cm, which gives $A_{\perp} = 2.7 \times 10^{-5} \text{ m}^2$. We take the drag coefficient to be $\mathcal{C} = 0.3$, a good approximation for a round-nosed elongated bullet. This signals better aerodynamics than the sphere's ($\mathcal{C}_D = 0.47$) but not as good as Toyota Prius' ($\mathcal{C}_D = 0.25$) [2]. Finally we are free to pick up some wind speed. In case of 20 mph wind (neither too weak not a hurricane), which in SI units translates into $u_{wind} = 9 \text{ m/s}$, we obtain

$$\Delta x = 6 \text{ cm.} \quad (7)$$

Is this displacement significant? Quite so. The diameter of the 10-ring on a free pistol target is 5 cm while that of the 9-ring is 10 cm, and so on, with the lower rings following the pattern. We see that the 20 mph wind can cost the shooter up to 3 points, as 6 cm is deflection large enough to move the point of bullet's impact from just the outside of the 10-ring to the inside of the 7-ring! Of course the wind's effect on the shooter's ability to hold steady and aim is probably going to be much more significant, as Olympic pistol shooting is performed one-handed in a rather precarious stance, unlike rifle shooting. On the other hand under such bad windy conditions the shooter would try to game the wind by waiting out for a periods of relative calm, but match tactics are beyond our objectives here. Returning to the physics of bullet's flight let us just appreciate the luck pistol shooters enjoy – the dependence on the distance to the target L is *quadratic*, which means that at longer ranges the wind drift would become significantly more important very quickly.

At the end discussion of some implicitly made approximations is in order. First, we should verify how well the assumption of the constancy of the bullet's velocity holds over relevant distances L . How quickly is the aerodynamic drag decelerating the projectile? The relatively weak wind drift (6) can be completely ignored now. The dynamics of the bullet in the direction towards the target is described by the 2nd Newton's law, $du/dt = -F/m$. Since the drag force F given by equation (3) does not depend *explicitly* on time (only implicitly via u), a delightful “trick” is going to be very helpful (as it is in many other problems involving differential equations). Change the variable from time t to the distance from the muzzle l , which amounts to transformation of the acceleration,

$du/dt = (du/dl)(dl/dt) = (du/dl)u$. Using this expression in the left-hand side of the Newton's law and substituting (3) into its right-hand side we obtain a simple differential equation (note the cancelation of one power of u in both sides of the equation):

$$\frac{du}{dl} = -\frac{\mathcal{C}_D \rho A_{\perp}}{2m} u, \quad (8)$$

whose solution is elementary,

$$u(l) = u_0 \exp\left(-\frac{\mathcal{C}_D \rho A_{\perp} l}{2m}\right). \quad (9)$$

By using the same bullet parameters here as we did in the rest of the above discussion we find that $u(50\text{m}) \approx 0.95u_0$, so that the relative decrease in the bullet's velocity over the distance to the target is only 5%. This rather insignificant correction does not affect much the accuracy of our estimates (6) and (7).

As a concluding note, we have also neglected some other less strong but nevertheless interesting physical effects. One of those is the *Coriolis force* induced by the Earth's rotation. Its effect on projectile motion is considered in many theoretical mechanics textbooks. If interested, you can follow the calculations and likely come to the conclusion that at distances of interest to pistol shooters the Coriolis effect is not of much concern. The *Magnus effect* is more intriguing. The interplay of cross-wind and bullet's own spin (induced by grooves in the barrel of the gun) leads to the asymmetry of the air flow: for a bullet spinning clock-wise the wind blowing from the left produces faster flow above the bullet and slower flow below it. At the most basic level in terms of the Bernoulli law, the slower/faster flow is characterized by higher/lower pressures. The bullet in our example will therefore be deflected upwards. (In the case of wind from the right and the same direction of the bullet's spin the deflection would of course be downward.) Magnus effect should depend on the viscosity, even in case of high Reynolds numbers. Indeed, in the complete absence of viscosity the air flow would not even “care” if the bullet is spinning or not. We may try to estimate the Magnus effect at one of our next Undergraduate seminars more quantitatively.

[1] This change is due to the fact that the total velocity increases to $(u_0^2 + u_{wind}^2)^{1/2}$. In addition the drag coefficient might have changed slightly for an oblique air flow. Both are minuscule effects which in any case would exceed the accuracy of our calculations.

[2] Curiously, the drag coefficient of a Formula 1 race car

could easily reach 0.7-0.8 or be even higher. The reason is that such cars are set up not to achieve the least drag but to provide significant amount of downward force in order to increase traction needed in tight turns.